Discrete Mathematics 312 (2012) 1367–1371

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Magic and antimagic H-decompositions

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ARTICLE INFO

Article history:
Received 26 February 2011
Received in revised form 15 November 2011
Accepted 29 November 2011
Available online 24 January 2012

Keywords:
Graph decompositions
Magic labelings

ABSTRACT

A decomposition of a graph G into isomorphic copies of a graph H is H-magic if there is a bijection $f : V(G) \cup E(G) \to \{0, 1, \ldots, |V(G)| + |E(G)| - 1\}$ such that the sum of labels of edges and vertices of each copy of H in the decomposition is constant. It is known that complete graphs do not admit K_2-magic decompositions for $n > 6$. By using the results on the sumset partition problem, we show that the complete graph K_{2m+1} admits T-magic decompositions by any graceful tree with m edges. We address analogous problems for complete bipartite graphs and for antimagic and (a, d)-antimagic decompositions.

1. Introduction

A *total labeling* of a graph G with n vertices and m edges is a bijection $f : V(G) \cup E(G) \to \{0, 1, \ldots, n+m-1\}$. For each subgraph $H \subset G$, we define the weight of H by f as

$$f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e).$$

The labeling f is said to be *magic* if every subgraph isomorphic to K_2 has the same weight, that is, there is a constant c such that, for every edge $e = xy$, $f(x) + f(y) + f(e) = c$. A graph is magic if it admits a magic labeling. This notion of magic graphs was introduced by Kotzig and Rosa [8], and the study of magic graphs has a large literature; see, e.g., the comprehensive survey of Gallian [4] or the book by Wallis [13] devoted to the subject. It was shown by Kotzig and Rosa that complete graphs are not magic for $n \geq 7$, while all complete bipartite graphs are magic. By answering a question posed by Erdős, Pikhurko showed that, in fact, a magic graph with n vertices can have at most $cn^2 + o(n^2)$ edges with $c = 0.489\ldots$; see also [9] for a related result.

Extensions of these notions to graphs different from K_2 (edges) were introduced in [5,10]. For a fixed graph H, a labeling f of G is said to be *H-magic* if all subgraphs of G isomorphic to H have the same weight. Among other results, it is shown in [5] that the complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic, while the complete graph K_{n+1} is not. Other examples of H-magic graphs with different choices of H can be found in the references given above or in Jeyanthi and Selvagopal [7].

One can ask for different properties of a total labeling f. The (total) labeling is said to be *antimagic* if the weights of subgraphs isomorphic to H are pairwise distinct. By further requiring that the weights form an arithmetic progression with difference d and first element a, the labeling is called *(a, d)-antimagic*, a notion introduced by Bača et al. [1]; see also [6,12], or the book of Bača and Miller [2], which contains a wealth of open problems on the subject.

In many of the results about H-magic or H-antimagic graphs the host graph G is required to have a unique H-decomposition, that is, a partition of its edge set $E(G) = E_1 \cup \cdots \cup E_m$ such that each E_i induces a subgraph of G isomorphic

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doi:10.1016/j.disc.2011.11.041

to H. This is certainly the case when $H = K_2$. The definition of an H-magic decomposition below is suggested by this observation. Recall that a family $\mathcal{H} = \{H_1, \ldots, H_k\}$ of subgraphs of G is an *H-decomposition* of G if all subgraphs are isomorphic to H and its edge sets partition the edge set of G. In this case, we write

$$G = H_1 \oplus \cdots \oplus H_k.$$

An H-decomposition $\mathcal{H} = \{H_1, \ldots, H_k\}$ of G is *magic* if there is a total labeling f of G such that $f(H_1) = \cdots = f(H_k)$. Similarly, $\mathcal{H}$ is *antimagic* if $f(H_1), \ldots, f(H_k)$ are pairwise distinct, and (a, d)-*antimagic* if these numbers form an arithmetic progression with difference d and first element a. Note that, when $H = K_2$, all these notions coincide with their original counterparts. We also note that a magic H-decomposition can be viewed as an (a, d)-antimagic one with $d = 0$. We will use this convention in our statements.

Recall that a *graceful* labeling of a graph H with m edges is an injection $g: V(H) \to \{0, 1, \ldots, m\}$ such that, when an edge $e = xy$ is assigned the label $|g(x) - g(y)|$, the resulting edge labels are pairwise distinct. Graceful labelings originated as a means to attack the conjecture of Ringel [11], which states that the complete graph K_{2m+1} can be decomposed into $2m + 1$ copies of a given tree. The Ringel–Kotzig conjecture states that all trees are graceful (this is also known as the *graceful labeling conjecture*). Our first result is the following.

Theorem 1. *Let T be a graceful tree with m edges. The complete graph K_{2m+1} admits an (a, d)-antimagic T-decomposition for some a and all even $0 \le d \le m + 1$. Moreover, if m is odd, then the same statement holds for every $0 \le d \le m + 1$.*

We also consider the following bipartite version. An α-labeling of a tree T with m edges and bipartition $\{A, B\}$ is a particular case of graceful labeling in which we additionally require that the difference labels $f(x) - f(y)$ for an edge $xy \in T$ are all nonnegative when $x \in B$.

Theorem 2. *Let T be a tree with m edges. If T admits an α-labeling, then the complete bipartite graph $K_{m,m}$ admits an (a, d)-antimagic T-decomposition for some a and all $0 \le d \le m$ with the same parity as m. Moreover, if m is odd, then the same statement holds for each $0 \le d \le m$.*

2. Sumset partitions

As in [5,10], the proofs of our main results are based on the use of sumset partitions. We recall in this section some useful facts on this concept; see also [5,10].

Let $a < b$ be integers. Throughout the paper we denote by $[a, b]$ the integer interval $\{i \in \mathbb{Z}: a \le i \le b\}$. We also write $x + [a, b] = \{x + i: i \in [a, b]\}$.

Given a set X of integers and a partition $\mathcal{P} = (X_1, \ldots, X_k)$ of X into k parts, we denote by

$$\Sigma(\mathcal{P}) = (\Sigma(X_1), \ldots, \Sigma(X_k))$$

the *sumset partition* of $\mathcal{P}$, where $\Sigma(Y) = \sum_{y \in Y} y$. We will always order the partition in such a way that the sequence of subset sums $\Sigma(X_1) \le \cdots \le \Sigma(X_k)$ is non-decreasing.

When all sets in $\mathcal{P}$ have the same cardinality, we say that $\mathcal{P}$ is an *equipartition* of X, or a k-equipartition if we want to stress the number k of sets in $\mathcal{P}$.

The next lemma is basically contained in [5, Lemma 4.1]. Its proof can be described in terms of the so-called Kotzig arrays (see, e.g., [3,14]), and the result can be derived from analogous statements in [14]. We include below a proof of the lemma along the lines of [5, Lemma 4.1] for the convenience of the reader.

Lemma 1. *Let $h > 1$ and k be two positive integers. There exists a k-equipartition $\mathcal{P}$ of $[1, hk]$ such that $\Sigma(\mathcal{P})$ is an arithmetic progression with length k and difference d for each $0 \le d \le h$ with the same parity as h. Moreover, if k is odd, then the same result holds for every $0 \le d \le h$.*

Proof. We identify the partition $\mathcal{P}$ with a coloring $c = (\alpha_1\alpha_2 \cdots \alpha_N)$, $N = hk$, where $\alpha_i = j$ if and only if $x_i \in X_j$. For example, the partition $\mathcal{P} = (\{0, 1, 2\}, \{3, 5, 8\}, \{4, 6, 7\})$ of $X = [0, 8]$ is identified with the coloring $c = (111232332)$. In the same vein, we write $\Sigma(c) = \Sigma(\mathcal{P})$, where the underlying set X will be clear from the context.

If every block of k consecutive elements of the form $\{x_{tk+1} < x_{tk+2} < \cdots < x_{(t+1)k}\} \subset X$, $t = 0, 1, \ldots, N/k - 1$, contains precisely one element from each set of $\mathcal{P}$, then we say that $\mathcal{P}$ is *well distributed*. For instance, $\{\{1, 4, 5\}, \{2, 3, 6\}\}$ is a well-distributed equipartition of [5], whereas $\{\{1, 2, 3\}, \{4, 5, 6\}\}$ is not (see, e.g., [5,10]).

When $\mathcal{P}$ is a well-distributed equipartition of X, the corresponding coloring consists of a concatenation of words with length k, each of which is a permutation in the symmetric group $Sym(k)$ of $[1, k]$. For $\sigma_1, \sigma_2 \in Sym(k)$, we write the concatenation of the two permutations as $\sigma_1 * \sigma_2$. We also write σ^{*n} for the concatenation of n copies of σ. For a permutation $\sigma \in Sym(k)$, we denote by $\bar{\sigma}$ its inversion. For example, for $X = [0, 8]$ and $\sigma = (123)$, the coloring $\sigma * \bar{\sigma}^{*2} = (123321321)$ denotes the equipartition of X into the three sets $(\{0, 5, 8\}, \{1, 4, 7\}, \{2, 3, 6\})$.

It is clear that, for every permutation $\sigma \in Sym(k)$ and every positive integer h, $\Sigma(\sigma^{*h})$ consists of an arithmetic progression with length k and difference h. In terms of Kotzig arrays, this corresponds to placing the elements of $[1, hk]$ in an array with h rows and k columns and taking the sets of the partition consisting of the columns of the array. Similarly, for each $0 \le i \le h$, $\Sigma(\sigma^{*i} * \bar{\sigma}^{*(h-i)})$ consists of an arithmetic progression with length k and difference $2i - h$. This proves the first part of the lemma.

Suppose now that $k = 2t + 1$ is odd. Consider the permutation

$$\pi(i) = \begin{cases} 2(t+1-i), & 1 \le i \le t \\ 4t + 3 - 2i, & t + 1 \le i \le 2t + 1. \end{cases}$$

For example, if $k = 7$, then $\pi = (6427531)$. Then $\Sigma(\iota * \pi)$, where ι denotes the identity permutation, is an arithmetic progression with difference one. More generally, $\iota^{*i} * \pi * \bar{\iota}^{*(h-i-1)}$ is a k-equipartition of $[1, hk]$ for which $\Sigma(\iota^{*i} * \pi * \bar{\iota}^{*(h-i-1)})$ is an arithmetic progression with length k and difference $d = 2i - h + 1$. This completes the proof of the second part of the statement. □

We note that the conclusions in Lemma 1 also hold when we replace the interval $[1, hk]$ by any integer translation $a + [1, hk]$. We also stress that the proof is constructive, so equipartitions with the claimed properties can be explicitly obtained.

3. Magic and antimagic decompositions of K_{2m+1}

In this section we prove Theorem 1.

Let f_0 be a graceful labeling of a tree T with m edges. We recall that a graceful labeling provides a cyclic decomposition of K_{2m+1} as follows. Since T has $m + 1$ vertices, we have $f_0(V(T)) = [0, m]$. Let $S = \{|f_0(x) - f_0(y)| : xy \in E(T)\}$ be the set of edge labels of f_0. Since f_0 is a graceful labeling, the elements in S are pairwise distinct integers. Consider the elements in S as residue classes modulo $2m+1$, and consider the Cayley graph $G = Cay(\mathbb{Z}_{2m+1}, S \cup (-S)\}$. Note that, since $S \cap (-S) = \emptyset$ and $|S| = m$, G is (isomorphic to) the complete graph K_{2m+1}. We think of this complete graph as being edge colored, the edge xy colored by the element $s \in S$ such that $x - y \in \{s, -s\}$.

Let T_0 be an embedding of T in K_{2m+1} obtained by placing each vertex v of T in the vertex $f_0(v)$ of G. Then the rotations $\phi_i(x) = x + i$ place $2m+1$ edge-disjoint copies of T, $T_0, T_1, \ldots, T_{2m}$ with $V(T_i) = \phi_i(V(T_0))$, which decompose the complete graph K_{2m+1}. The sets $\{E(T_0), \ldots, E(T_{2m})\}$ form a partition of $E(K_{2m+1})$ because each rotation ϕ_i preserves the colors of the edges, and $E(T_0)$ contains precisely one edge of each color.

We now go back to the arithmetic in the integers and define a total labeling f_1 of K_{2m+1} for which the given T-decomposition of K_{2m+1} is magic.

We define $f_1(v) = 2v \pmod{2m+1}$ for every $v \in V(K_{2m+1})$. Observe that, in the integers, the sequence $\{\Sigma(V(T_i)), i = 0, 1, \ldots, m\}$ forms an arithmetic progression with difference one starting at $\Sigma(V(T_0)) = 0 + 2 + 4 + \cdots + 2m = m^2 + m$, and the sequence $\{\Sigma(V(T_i)), i = m+1, m+2, \ldots, 2m\}$ forms an arithmetic progression with difference one starting at $\Sigma(V(T_{m+1})) = 1 + 3 + 5 + \cdots + 2m - 1 = m^2$. Therefore the whole sequence

$$\{\Sigma(V(T_0)), \Sigma(V(T_1)), \ldots, \Sigma(V(T_{2m}))\} = m^2 + [0, 2m],$$

forms an arithmetic progression with difference one.

The labeling of the set of edges must take its values in the interval

$$I_m = 2m + [1, m(2m+1)].$$

Since the edge sets $E(T_0), \ldots, E(T_{2m})$ are pairwise disjoint and f_1 must be a bijection from $E(K_{2m+1})$ to the interval I_m, the family of sets $\{f_1(E(T_0)), \ldots, f_1(E(T_{2m}))\}$ will form a partition of the interval I_m.

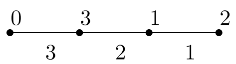
By Lemma 1, for each odd $0 \le d \le m$, if m is even and each $0 \le d \le m$ if m is odd, there exists a $(2m+1)$-equipartition $\mathcal{P} = \{X_1, \ldots, X_{2m+1}\}$ of I_m such that $\Sigma(\mathcal{P})$ forms an arithmetic progression with length $2m + 1$ and difference d. We may assume that $\Sigma(X_1) \le \Sigma(X_2) \le \cdots \le \Sigma(X_{2m+1})$. For a permutation $\sigma \in Sym(2m+1)$, let us consider the sequence

$$\{\Sigma(V(T_i)) + \Sigma(X_{\sigma(i+1)}) : 0 \le i \le 2m\}. \tag{1}$$

Since $\{\Sigma(V(T_0)), \Sigma(V(T_1)), \ldots, \Sigma(V(T_{2m}))\} = m^2 + [0, 2m]$ is an arithmetic progression with difference one, with the choice $\sigma(i+1) = i+1$ (respectively, $\sigma(i+1) = 2m+1-i$) for each i, the sequence (1) forms an arithmetic progression with difference $d + 1$ or $d - 1$, respectively.

We can define f_1 on $E(T_i)$ as any bijection to the set $X_{\sigma(i+1)}$ of $\mathcal{P}$. By doing so, we obtain total labelings of K_{2m+1} such that our T-decomposition is (a, d)-antimagic for some a and each even $0 \le d \le m+1$, or for each $0 \le d \le m+1$ if m is odd. This completes the proof of the theorem. □

Fig. 1 illustrates the decomposition of K_7 by the path P_3 with three edges and the corresponding magic labeling. For this, we use the graceful labeling of the path P_3 shown below:



We embed P_3 in K_7 by multiplying the labels of vertices by two. The seven rotations of this embedding provide a decomposition of K_7 by copies of P_3. The 7-equipartition of $I_3 = 6 + [1, 21]$ is given, according to the proof of Lemma 1, as

$$\mathcal{P} = \{\{7, 14, 27\}, \{8, 15, 26\}, \{9, 16, 25\}, \{10, 17, 24\}, \{11, 18, 23\}, \{12, 19, 22\}, \{13, 21, 20\}\}.$$

By appropriately matching the edge sets of the copies of P_3 with the sets in $\mathcal{P}$, we obtain the magic P_3-decomposition of K_7 displayed in Fig. 1.

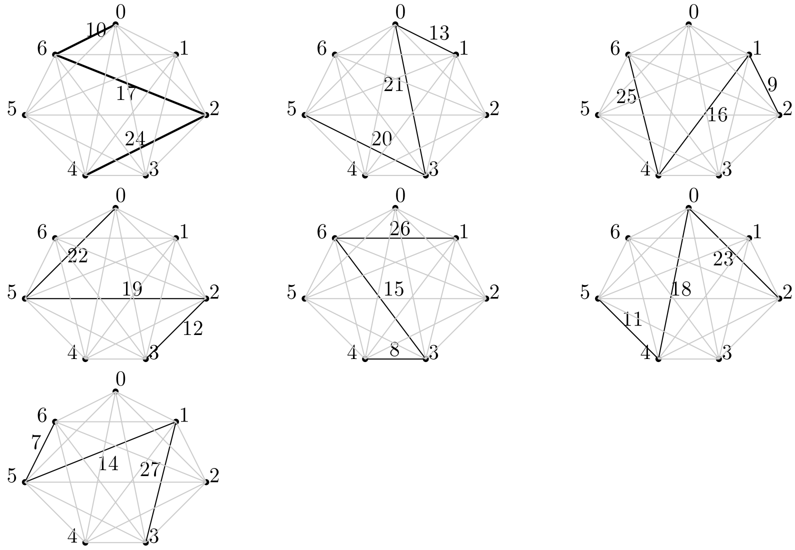


Fig. 1. An example of the magic decomposition of K_7 by the path P_3 with common weight 63.

4. Magic and antimagic decompositions of $K_{m,m}$

In this section, we prove Theorem 2.

Let T be a tree which admits an α-labeling $f_0 : V(T) \to [0, m]$. Let us recall that such a labeling provides a cyclic decomposition of the bipartite complete graph $K_{m,m}$. Denote by A and B the two color classes of vertices of the tree T such that $f_0(A) = [0, |A| - 1]$ and $f_0(B) = [|A|, m]$. Consider the map $f : V(T) \to \mathbb{Z}_m \times \mathbb{Z}_2$, defined as

$$f(v) = \begin{cases} (f_0(v) (\bmod m), 0), & v \in A; \\ (f_0(v) (\bmod m), 1), & v \in B. \end{cases}$$

Observe that, for every pair of edges $xy, x'y' \in E(T)$ with $y, y' \in A$, if $f(x) - f(y) = f(x') - f(y')$, then we must have $f_0(x) - f_0(y) \equiv f_0(x') - f_0(y') (\bmod m)$. Since f_0 is an α-labeling, the differences of labels by f_0 belong to $[1, m]$, so the above congruence implies in fact that $f_0(x) - f_0(y) = f_0(x') - f_0(y')$, and therefore $xy = x'y'$. Let $S = \{f(x) - f(y), xy \in E(T), y \in A\}$. By our previous remark, we have $S = \mathbb{Z}_m \times \{1\}$. Therefore, the underlying graph of the directed Cayley graph $G = Cay(\mathbb{Z}_m \times \mathbb{Z}_2, S)$ is the complete bipartite graph $K_{m,m}$. As usual, we think of the edge $(x, 0)(y, 1)$ as being colored by $y - x$. Thus the map f is an embedding of the directed tree, obtained from T by orienting all arcs from vertices in A to vertices in B into G. By this embedding, no two arcs of T have the same color. Therefore, the set

$$\{T_0, \phi(T_0), \ldots, \phi^{m-1}(T_0)\},$$

where $T_0 = f(T)$ and $\phi(x, i) = (x + 1, i)$ for each $(x, i) \in \mathbb{Z}_m \times \mathbb{Z}_2$, is a T-decomposition of $K_{m,m}$ when we ignore the orientation of the edges.

We consider two cases.

Case 1: m is even.

Let us label the partite sets of $K_{m,m}$, one with the even integers $\{0, 2, \ldots, 2m - 2\}$ and the other with the odd integers $\{1, 3, \ldots, 2m - 1\}$. Define the embedding $f_1 : V(T) \to V(K_{m,m})$ as $f_1(v) = 2 \cdot \overline{f_0(v)}$ if $v \in A$ and $f_1(v) = 2 \cdot \overline{f_0(v)} + 1$ if $v \in B$, where $\bar{x}$ denotes the representative of x modulo m in $[0, m - 1]$. It follows from the arguments above that the family of trees

$$\{T_0, \varphi(T_0), \ldots, \varphi^{m-1}(T_0)\},$$

where now $T_0 = f_1(T)$ and $\varphi(x)$ is the representative of $x + 2 (\bmod 2m)$ in $[0, 2m - 1]$, forms a T-decomposition of $K_{m,m}$.

Denote by $T_i = \varphi^i(T_0)$ and observe that, for $i = 1, \ldots, m - 1$, we have

$$\Sigma(V(T_i)) = \Sigma(V(T_{i-1})) + 2,$$

since the labels of all vertices increase their value by 2, except precisely one label, which either changes from $2m - 2$ to 0 or from $2m - 1$ to 1. Therefore, the sequence $\{\Sigma(V(T_i)), i = 0, \ldots, m - 1\}$ forms an arithmetic progression with difference two.

We now extend the labeling f_1 to the set of edges of $K_{m,m}$. The edges must be labeled with the integers in the interval $I_m = 2m - 1 + [1, m^2]$. For each even d with $0 \le d \le m$, consider the m-equipartition $\mathcal{P} = \{X_1, \ldots, X_m\}$ of I_m given by Lemma 1 whose sequence of subset sums $\Sigma(\mathcal{P})$ is also an arithmetic progression with difference d. By defining a bijection from the sets $E(T_i)$ to the appropriate sets X_j, we obtain a total labeling of $K_{m,m}$ for which our T-decomposition is an arithmetic progression with difference $d \pm 2$.

Case 2: m is odd.

Let us now label the partite sets of $K_{m,m}$, one with the integers $\{0, 1, \dots, m-1\}$ and the other with the integers $\{m, m+1, \dots, 2m-1\}$. Define the embedding $f_1 \colon V(T) \to V(K_{m,m})$ as $f_1(v) = \overline{f_0(v)}$ if $v \in A$ and $f_1(v) = m + \overline{f_0(v)}$ if $v \in B$, where $\bar{x}$ denotes the representative of x modulo m in $[0, m-1]$. It follows from the arguments above that the family of trees

$$\{T_0, \varphi(T_0), \dots, \varphi^{m-1}(T_0)\},$$

where now $T_0 = f_1(T)$ and $\varphi(x)$ is the representative of $x + 1 (\mathrm{mod}\ 2m)$ in $[0, m-1]$ if $x \in A$, and the representative of $x + 1 (\mathrm{mod}\ 2m)$ in $[m, 2m-1]$ if $x \in B$, forms a T-decomposition of $K_{m,m}$. Observe that the sequence $\{\Sigma(V(T_i)), i = 0, \dots, m-1\}$ forms an arithmetic progression with difference one.

We now extend the labeling f_1 to the set of edges of $K_{m,m}$. The edges must be labeled with the integers in the interval $I_m = 2m - 1 + [1, m^2]$. For each $0 \le d \le m$, consider the m-equipartition $\mathcal{P} = \{X_1, \dots, X_m\}$ of the interval I_m given by Lemma 1 whose sequence of subset sums $\Sigma(\mathcal{P})$ is an arithmetic progression with difference d. By defining a bijection from the sets $E(T_i)$ to the appropriate sets X_j, we obtain a total labeling of $K_{m,m}$ for which our T-decomposition is an arithmetic progression with difference $d \pm 1$. This completes the proof of Theorem 2.

Acknowledgments

The authors are grateful for the valuable comments of the anonymous referees and for calling our attention to [3]. This work was supported by the Spanish Research Council under project MTM2008-06620-C03-01 and the Catalan Research Council under project 2009SGR01387.

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